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Oscillation of certain second order nonlinear damped difference equations with continuous variable[☆]

Peiguang Wang[a,*], Meng Wu[b]

[a] College of Electronic and Information Engineering, Hebei University, Baoding, 071002, China
[b] College of Mathematics and Computer Science, Hebei University, Baoding, 071002, China

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Abstract

In this work, a class of second order nonlinear damped difference equations with continuous variable are investigated. Some oscillatory criteria are obtained.

Keywords: Continuous variable; Damping term; Oscillation; Generalized Riccati transformation; Integral averaging technique

1. Introduction

In this work, we consider the oscillation behavior of solutions of the following second order nonlinear damped difference equation with continuous variable:

$$\Delta_\tau^2 x(t) + p(t)\Delta_\tau x(t) + q(t) f(t, x(t - \sigma)) = 0, \tag{1}$$

where τ and σ are nonnegative constants, $\Delta_\tau x(t) = x(t + \tau) - x(t)$, $t \in I = [t_0, \infty)$.

Throughout this work, we always assume that the following conditions hold.

(H_1) $p \in C(I, (0, 1))$, $q \in C(I, R^+)$ and $q(t) \not\equiv 0$ on any ray $[t_\mu, \infty)$ for some $t_\mu \geq t_0$;
(H_2) $f \in C(I \times R, R)$, $f(t, 0) \equiv 0$ and $\frac{f(t,x)}{x} \geq K > 0$ for all $x \neq 0$.

We restrict our attention to solutions $x(t)$ of Eq. (1) which exist on some half-line $[t_0, \infty)$ and are nontrivial for all sufficiently large t, that is, satisfy $\sup\{|x(t)|, t \geq t_1\} > 0$ for any $t_1 \geq t_0$. It is tacitly assumed that such solutions exist.

As is customary, a solution $x(t)$ of Eq. (1) is said to be eventually positive (negative) if there exists a sufficiently large positive number t_μ such that the inequality $x(t) > 0\,(x(t) < 0)$ holds for $t \geq t_\mu$. A solution $x(t)$ of Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

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* Corresponding author.
E-mail address: pgwang@mail.hbu.edu.cn (P. Wang).

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Oscillatory behaviors of second order nonlinear difference equations with damping have been obtained by many authors (see [1–3] and the references quoted therein). Among these, Thandapani et al. [1] considered the second order damped difference equation

$$\Delta(a_n \Delta y_n) + p_n \Delta y_n + q_n f(y(\sigma(n+1))) = 0, \quad n = 0, 1, 2 \dots. \tag{2}$$

When $\sigma(n+1) = n$, (2) can be thought of as a discrete analogue of

$$(a(t)y'(t))' + p(t)y'(t) + q(t)f(y(t)) = 0.$$

Recently, there has been increasing interest in the study of the oscillation of difference equations with continuous variable. Zhang et al. [4] considered the second order nonlinear difference equation

$$\Delta_\tau^2 x(t) + f(t, x(t-\sigma)) = 0, \tag{3}$$

and obtained oscillatory criteria under some conditions. However, to the best of our knowledge, very little has been done with the case of second order damped difference equations with continuous variable. Motivated by the ideas of [5–7], in this work, we introduce parameter functions and integral operators, and obtain new oscillatory criteria by using a generalized Riccati transformation and integral averaging technique. These results are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line.

To prove our main results, we need the following lemmas.

Lemma 1.1. *Let $x(t)$ be a nonoscillatory solution of Eq. (1). If for sufficiently large t*

(H_3)

$$\limsup_{n\to\infty} \sum_{i=0}^{n} \prod_{j=0}^{i-1} [1 - p(t + j\tau)] = \infty$$

holds, then there exists $t_1 \geq t_0$ such that $x(t)\Delta_\tau x(t) > 0$, for all $t \geq t_1$.

Proof. Without loss of generality, let $x(t)$ be an eventually positive solution (the result for $x(t)$ an eventually negative solution can be similarly proved). Therefore, there exists $T \geq t_0$ such that $x(t) > 0$, $x(t-\sigma) > 0$, $t \geq T$. We claim that $\Delta_\tau x(t)$ is eventually positive. Indeed, suppose, to the contrary, there exists $t_1 \geq T \geq t_0$ such that

$$\Delta_\tau x(t_1) < 0 \quad \text{or} \quad \Delta_\tau x(t_1) = 0.$$

Considering $\Delta_\tau x(t_1) < 0$, in view of Eq. (1), we have

$$\begin{aligned}\Delta_\tau^2 x(t_1) &= -p(t_1)\Delta_\tau x(t_1) - q(t_1) f(t_1, x(t_1 - \sigma)) \\ &\leq -p(t_1)\Delta_\tau x(t_1) - Kq(t_1)x(t_1 - \sigma) \leq -p(t_1)\Delta_\tau x(t_1).\end{aligned}$$

Hence

$$\Delta_\tau x(t_1 + \tau) - \Delta_\tau x(t_1) \leq -p(t_1)\Delta_\tau x(t_1),$$

that is

$$\Delta_\tau x(t_1 + \tau) \leq [1 - p(t_1)]\Delta_\tau x(t_1).$$

By (H_1) and $\Delta_\tau x(t_1) < 0$, we have $\Delta_\tau x(t_1 + \tau) < 0$. Next, assume that $\Delta_\tau x(t_1 + (k-1)\tau) < 0$, $k \in N$, holds. Repeating the above process, we obtain $\Delta_\tau x(t_1 + k\tau) < 0$. By mathematical induction, $\Delta_\tau x(t_1 + n\tau) < 0$ holds for all positive integers n. Hence we obtain $\Delta_\tau x(t) < 0$, for all $t \geq t_1$.

Considering $\Delta_\tau x(t_1) = 0$, in view of Eq. (1), we have

$$\Delta_\tau^2 x(t_1) = \Delta_\tau x(t_1 + \tau) - \Delta_\tau x(t_1) = -q(t_1) f(t_1, x(t_1 - \sigma)) \leq 0,$$

which implies that $\Delta_\tau x(t_1 + \tau) \leq 0$. If $\Delta_\tau x(t_1 + \tau) < 0$, then, on the basis of what we have just observed, we may conclude that $\Delta_\tau x(t) < 0$, $t \geq t_1 + \tau$. If $\Delta_\tau x(t_1 + \tau) = 0$, we may conclude that $\Delta_\tau x(t_1 + 2\tau) \leq 0$. Repeating the above process, we will end up with two situations:

$$\Delta_\tau x(t) < 0 \quad \text{or} \quad \Delta_\tau x(t) = 0, \quad t \geq t_1.$$

However, $\Delta_\tau x(t) = 0$, $t \geq t_1$, is impossible. Indeed, if there exists $T_1 > t_1$ such that $q(T_1) > 0$, then in view of Eq. (1), we have

$$0 = \Delta_\tau^2 x(T_1) + p(T_1)\Delta_\tau x(T_1) + q(T_1) f(T_1, x(T_1 - \sigma)) = q(T_1) f(T_1, x(T_1 - \sigma)) > 0,$$

which is a contradiction.

If we define $u(t) = -\Delta_\tau x(t)$ for $t \geq t_1$ such that $\Delta_\tau x(t) < 0$, then from Eq. (1), we have

$$\Delta_\tau u(t) + p(t)u(t) \geq 0,$$

and hence

$$u(t + \tau) \geq [1 - p(t)]u(t).$$

Thus

$$u(t + i\tau) \geq u(t) \prod_{j=0}^{i-1} [1 - p(t + j\tau)], \quad i = 0, 1, 2 \ldots,$$

so

$$\Delta_\tau x(t + i\tau) \leq \Delta_\tau x(t) \prod_{j=0}^{i-1} [1 - p(t + j\tau)].$$

Summing the last inequality from 0 to $k - 1$, we have

$$x(t + k\tau) - x(t) \leq \Delta_\tau x(t) \sum_{i=0}^{k-1} \prod_{j=0}^{i-1} [1 - p(t + j\tau)].$$

When $k \to \infty$, taking the superior limit, condition (H_3) implies that $x(t)$ is eventually negative, which is a contradiction. The proof is complete. □

In the sequel we say that a function $H = H(t, s)$ belongs to a function class E, denoted by $H \in E$, if $H \in C(D, R^+)$, where $D = \{(t, s) : -\infty < s \leq t < \infty\}$, which satisfies

$$H(t, t) = 0, \qquad H(t, s) > 0, \quad \text{for } t > s,$$

and has partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t, s)H(t, s) \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t, s)H(t, s),$$

where $h_1, h_2 \in L_{\text{loc}}(D, R)$.

Let $k \in C'(I, R^+)$ and $z \in C(I, R)$, for $t \geq T \geq t_0$; we now define two integral operators X, $Y \in C(R, R)$ and

$$X_{T,t}^{H,k}(z) = \int_T^t H(s, T)k(s)z(s)\mathrm{d}s \quad \text{and} \quad Y_{T,t}^{H,k}(z) = \int_T^t H(t, s)k(s)z(s)\mathrm{d}s.$$

In order to discuss our results, we assume throughout this work that $g \in C'(I, R)$ and $\rho(t) = \exp[-2\int^t g(s)\mathrm{d}s]$. Define

$$\bar{p}(t) = \min_{t \leq s \leq t+2\tau} \left\{ \frac{p(s)}{2\tau} \right\}, \qquad \bar{q}(t) = \min_{t \leq s \leq t+2\tau} \left\{ \frac{q(s)}{\tau^2} \right\},$$

$$\theta(t) = \rho(t)[-\bar{p}(t)g(t) + K\bar{q}(t) + g^2(t) - g'(t)], \qquad \varphi(t) = -\bar{p}(t). \tag{4}$$

Lemma 1.2. *Assume that* (H_3) *holds. Let $x(t)$ be an eventually positive solution of Eq.* (1) *on* $(a, c]$. *Then there exists a function $w \in C'(I, R)$ such that*

$$X_{a,c}^{H,k} \left[\theta - \frac{1}{4}\rho \left(h_1 + \varphi + \frac{k'}{k} \right)^2 \right] \leq -H(c, a)k(c)w(c) \tag{5}$$

holds for any $H \in E$, where θ and φ are defined in (4).

Proof. Because $x(t)$ is an eventually positive solution of Eq. (1), in view of Lemma 1.1, we have $\Delta_\tau x(t) > 0$, $t \geq t_1$. As in [4], letting

$$u(t) = \int_t^{t+\tau} \mathrm{d}s \int_s^{s+\tau} x(\theta)\mathrm{d}\theta,$$

then $u(t) > 0$, $u''(t) = \Delta_\tau^2 x(t) \leq 0$, $u'(t) > 0$, and Eq. (1) becomes

$$u''(t) + p(t)\Delta_\tau x(t) + Kq(t)x(t-\sigma) \leq 0, \quad t \geq t_1. \tag{6}$$

Integrating (6) in the region

$$G = \begin{cases} s \leq \theta \leq s+\tau, \\ t \leq s \leq t+\tau, \end{cases}$$

we have

$$\Delta_\tau^2 u(t) + p_0(t)\Delta_\tau u(t) + Kq_0(t)u(t-\sigma) \leq 0, \tag{7}$$

where $p_0(t) = \min_{t \leq s \leq t+2\tau} \{p(s)\}$, $q_0(t) = \min_{t \leq s \leq t+2\tau} \{q(s)\}$. Define

$$v(t) = \int_t^{t+\tau} \mathrm{d}s \int_s^{s+\tau} u(\theta)\mathrm{d}\theta.$$

Since $u'(t) > 0$, we have $v(t) \leq \tau^2 u(t+2\tau)$, $v(t-\sigma-2\tau) \leq \tau^2 u(t-\sigma)$, $v''(t) = \Delta_\tau^2 u(t) \leq 0$, i.e. $v'(t) \leq v'(t-\sigma-2\tau)$ and

$$\begin{aligned} v'(t) &= \int_{t+\tau}^{t+2\tau} u(\theta)\mathrm{d}\theta - \int_t^{t+\tau} u(\theta)\mathrm{d}\theta \\ &\leq \tau u(t+2\tau) - \tau u(t) = \tau\left[\Delta_\tau u(t+\tau) + \Delta_\tau u(t)\right] \leq 2\tau\Delta_\tau u(t). \end{aligned}$$

Therefore, (7) becomes

$$v''(t) + \frac{p_0(t)}{2\tau}v'(t) + \frac{Kq_0(t)}{\tau^2}v(t-\sigma-2\tau) \leq 0,$$

i.e.

$$v''(t) + \bar{p}(t)v'(t) + K\bar{q}(t)v(t-\sigma-2\tau) \leq 0.$$

Letting

$$w(t) = \rho(t)\left[\frac{v'(t)}{v(t-\sigma-2\tau)} + g(t)\right],$$

then

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\left[\frac{v''(t)}{v(t-\sigma-2\tau)} - \frac{v'(t)v'(t-\sigma-2\tau)}{v^2(t-\sigma-2\tau)} + g'(t)\right] \\ &\leq \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\left\{-\bar{p}(t)\frac{v'(t)}{v(t-\sigma-2\tau)} - K\bar{q}(t) - \left[\frac{v'(t)}{v(t-\sigma-2\tau)}\right]^2 + g'(t)\right\} \\ &= \left[\frac{\rho'(t)}{\rho(t)} - \bar{p}(t) + 2g(t)\right]w(t) - \frac{w^2(t)}{\rho(t)} - \rho(t)[-\bar{p}(t)g(t) + K\bar{q}(t) + g^2(t) - g'(t)] \\ &= \varphi(t)w(t) - \frac{w^2(t)}{\rho(t)} - \theta(t). \end{aligned}$$

So for all $t \geq t_0$, we have

$$\theta(t) \leq -w'(t) + \varphi(t)w(t) - \frac{w^2(t)}{\rho(t)}. \tag{8}$$

Applying the operator $X_{t,c}^{H,k}$ to (8) on $(a, c]$, we get

$$\begin{aligned}
X_{t,c}^{H,k}(\theta) &\leq -\int_t^c H(s,t)k(s)w'(s)\mathrm{d}s + \int_t^c H(s,t)k(s)\varphi(s)w(s)\mathrm{d}s - \int_t^c H(s,t)k(s)\frac{w^2(s)}{\rho(s)}\mathrm{d}s \\
&= -H(c,t)k(c)w(c) + X_{t,c}^{H,k}(h_1 w) + X_{t,c}^{H,k}\left(\frac{k'}{k}w\right) + X_{t,c}^{H,k}(\varphi w) - X_{t,c}^{H,k}\left(\frac{w^2}{\rho}\right) \\
&= -H(c,t)k(c)w(c) + X_{t,c}^{H,k}\left[\left(h_1 + \varphi + \frac{k'}{k}\right)w - \frac{w^2}{\rho}\right] \\
&= -H(c,t)k(c)w(c) - X_{t,c}^{H,k}\left\{\frac{1}{\rho}\left[w - \frac{1}{2}\rho\left(h_1 + \varphi + \frac{k'}{k}\right)\right]^2\right\} + \frac{1}{4}X_{t,c}^{H,k}\left[\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right] \\
&\leq -H(c,t)k(c)w(c) + \frac{1}{4}X_{t,c}^{H,k}\left[\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right].
\end{aligned}$$

That is,

$$X_{t,c}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right] \leq -H(c,t)k(c)w(c).$$

Letting $t \to a^+$ in the above, we obtain (5). The proof is complete. □

Lemma 1.3. *Assume that* (H$_3$) *holds. Let $x(t)$ be an eventually positive solution of Eq.* (1) *on* $[c, b)$. *Then there exists a function $w \in C'(I, R)$ such that*

$$Y_{c,b}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_2 - \varphi - \frac{k'}{k}\right)^2\right] \leq H(b,c)k(c)w(c) \tag{9}$$

holds for any $H \in E$, where θ and φ are defined in (4).

Proof. Like in the proof of Lemma 1.2, applying the operator $Y_{c,t}^{H,k}$ to (8) on $[c, b)$, we have

$$Y_{c,t}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_2 - \varphi - \frac{k'}{k}\right)^2\right] \leq H(t,c)k(c)w(c).$$

Letting $t \to b^-$ in the above, we obtain (9). The proof is complete. □

2. Main results

The following theorems are immediate results from Lemmas 1.1–1.3.

Theorem 2.1. *Assume that* (H$_3$) *holds. For some $H \in E$, $k, \rho \in C'(I, R^+)$ and $g \in C'(I, R)$, and for each $l \geq t_0$, assume that*

$$\limsup_{t\to\infty} X_{l,t}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right] > 0, \tag{10}$$

and

$$\limsup_{t\to\infty} Y_{l,t}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_2 - \varphi - \frac{k'}{k}\right)^2\right] > 0. \tag{11}$$

hold. Then Eq. (1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq. (1). Without loss of generality, let $x(t)$ be an eventually positive solution (the result for $x(t)$ an eventually negative solution can be similarly proved). For any $T \geq t_0$, let $a = T$. In (10), we choose $l = a$. Then there exists $c > a$ such that

$$X_{a,c}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right] > 0. \tag{12}$$

In (11), we choose $l = c$. Then there exists $b > c$ such that

$$Y_{c,b}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_2 - \varphi - \frac{k'}{k}\right)^2\right] > 0. \tag{13}$$

Combining (12) and (13), we obtain

$$\frac{1}{H(c,a)}X_{a,c}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right] + \frac{1}{H(b,c)}Y_{c,b}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_2 - \varphi - \frac{k'}{k}\right)^2\right] > 0. \tag{14}$$

By dividing (5) and (9) by $H(c,a)$ and $H(b,c)$, respectively, and then adding them, we have that

$$\frac{1}{H(c,a)}X_{a,c}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_1 + \varphi + \frac{k'}{k}\right)^2\right] + \frac{1}{H(b,c)}Y_{c,b}^{H,k}\left[\theta - \frac{1}{4}\rho\left(h_2 - \varphi - \frac{k'}{k}\right)^2\right] \leq 0,$$

which contradicts (14). The proof is complete. □

From the above oscillation criteria, we can obtain different sufficient conditions for all solutions of Eq. (1) by making different choices of $H(t,s)$. For example, let

$$H(t,s) = (t-s)^n, \quad t \geq s \geq t_0,$$

where $n > 1$ is a constant. Then $h_1 = h_2 = n(t-s)^{-1}$. Thus from Theorem 2.1, we have

Corollary 2.1. *Assume that* (H$_3$) *holds and* $k(t) = 1$, $t \geq t_0$. *If there exists a function* $\rho \in C'(I, R^+)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_l^t (s-l)^n \left\{\theta(s) - \frac{\rho(s)}{4}\left[\frac{n}{s-l} - \bar{p}(s)\right]^2\right\} \mathrm{d}s > 0, \tag{15}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_l^t (t-s)^n \left\{\theta(s) - \frac{\rho(s)}{4}\left[\frac{n}{t-s} + \bar{p}(s)\right]^2\right\} \mathrm{d}s > 0 \tag{16}$$

hold for each $l \geq t_0$ *and some* $n > 1$, *then Eq.* (1) *is oscillatory.*

Example 2.1. Consider the nonlinear difference equation

$$\triangle[x(t+1) - x(t)] + \frac{1}{t-2}[x(t+1) - x(t)] + \frac{1}{t-2}x(t-\sigma)[1 + x^2(t-\sigma)] = 0, \tag{17}$$

where $t > 3$. It is obvious that $\bar{p}(t) = \frac{1}{2t}$, $\bar{q}(t) = \frac{1}{t}$, and $\frac{f(t,x)}{x} \geq 1$, for all $x \neq 0$. We can say that (4) holds, because

$$\begin{aligned}\limsup_{n\to\infty} \sum_{i=0}^{n} \prod_{j=0}^{i-1} [1 - p(t + j\tau)] &= \limsup_{n\to\infty} \sum_{i=0}^{n} \left[\left(1 - \frac{1}{t-2}\right)\left(1 - \frac{1}{t-1}\right)\cdots\left(1 - \frac{1}{t-3+i}\right)\right] \\ &= \limsup_{n\to\infty} \sum_{i=0}^{n} \frac{t-3}{t-3+i} = \infty.\end{aligned}$$

Let $g(t) = -\frac{1}{2t}$, and $n = 2$. Thus $\rho(t) = t$, and $\theta(t) = 1$. Then we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_l^t (s-l)^2 \left\{\theta(s) - \frac{\rho(s)}{4}\left[\frac{n}{s-l} - \bar{p}(s)\right]^2\right\} \mathrm{d}s$$
$$= \limsup_{t\to\infty} \frac{1}{t}\int_l^t (s-l)^2 \left[1 - \frac{s}{4}\left(\frac{2}{s-l} - \frac{1}{2s}\right)^2\right] \mathrm{d}s = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_l^t (t-s)^2 \left\{\theta(s) - \frac{\rho(s)}{4}\left[\frac{2}{t-s} + \bar{p}(s)\right]^2\right\} \mathrm{d}s$$
$$= \lim_{t\to\infty} \frac{1}{t}\int_l^t (t-s)^2 \left[1 - \frac{s}{4}\left(\frac{2}{t-s} + \frac{1}{2s}\right)^2\right] \mathrm{d}s = \infty.$$

Thereby, Eq. (17) is oscillatory by Corollary 2.1.

For the case where $H := H(t-s) \in E$, we have that $h_1(t-s) = h_2(t-s)$, and we denote them as $h(t-s)$. The subclass of E containing such $H(t-s)$ is denoted by E_0. We obtain

Theorem 2.2. *Assume that* (H_3) *holds. If there exist a function* $H \in E_0$ *and* $\rho \in C'(I, R^+)$ *such that*

$$\int_a^c H(s-a)[k(s)\theta(s) + k(2c-s)\theta(2c-s)]\mathrm{d}s$$
$$> \frac{1}{4}\int_a^c H(s-a)[k(s)\rho(s) + k(2c-s)\rho(2c-s)]h^2(s-a)\mathrm{d}s, \tag{18}$$

holds, where $k(s) = \exp[\int^t -\varphi(u)\mathrm{d}u]$, $c > a \ge T \ge t_0$, *then Eq.* (1) *is oscillatory.*

Proof. Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H(\frac{b-a}{2})$, and for any $f \in L[a, b]$, we have

$$\int_c^b f(s)\mathrm{d}s = \int_a^c f(2c-s)\mathrm{d}s.$$

Note that $\varphi(t) + \frac{\rho'(t)}{\rho(t)} = 0$; then we get

$$\int_c^b H(b-s)k(s)\theta(s)\mathrm{d}s = \int_a^c H(s-a)k(2c-s)\theta(2c-s)\mathrm{d}s,$$

and

$$\int_c^b H(b-s)k(s)\rho(s)h(s)\mathrm{d}s = \int_a^c H(s-a)k(2c-s)\rho(2c-s)h(2c-s)\mathrm{d}s.$$

Thus that (18) holds implies that (14) holds, and therefore every solution of Eq. (1) is oscillatory by Theorem 2.1. The proof is complete. □

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